Student number: ________________

GIRRAWEEN HIGH SCHOOL

2022

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt all questions
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. Given that α and β are both acute angles, evaluate $\sin(\alpha + \beta)$ if

   $\sin \alpha = \dfrac{8}{17}$ and $\sin \beta = \dfrac{4}{5}$.

   A. $\dfrac{108}{85}$ B. $\dfrac{84}{85}$ C. $\dfrac{36}{85}$ D. $\dfrac{28}{85}$

2. If $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, the unit vector in the direction of $\underset{\sim}{u}$ is:

   A. $\dfrac{1}{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

   B. $\dfrac{1}{5}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

   C. $\sqrt{13}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

   D. $\dfrac{1}{\sqrt{13}}\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

3. A polynomial $P(x)$ has a triple root at $x = -2$.
   Which of the following statements is true?

   A. $(x + 2)^2$ is a factor of $P'(x)$.

   B. $(x - 2)^2$ is a factor of $P'(x)$.

   C. $(x + 2)^3$ is a factor of $P'(x)$.

   D. $(x - 2)^3$ is a factor of $P'(x)$.

4. Which of the following is an expression for $\int \sin^2 4x \, dx$?

A. $\frac{x}{2} - \frac{1}{16} \sin 8x + c$

B. $\frac{x}{2} + \frac{1}{16} \sin 8x + c$

C. $x - \frac{1}{8} \sin 8x + c$

D. $x + \frac{1}{8} \sin 8x + c$

5. The equation $x^3 - 2x^2 - 4x + 8 = 0$ has roots α, β and γ.

What is the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$?

A. $-\frac{1}{2}$ B. $-\frac{1}{4}$ C. $\frac{1}{4}$ D. $\frac{1}{2}$

6. The domain and range of $y = 4 \cos^{-1}\left(\frac{3x}{2}\right)$ is

A. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$

Range: $-2\pi \leq y \leq 2\pi$

B. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$

Range: $0 \leq y \leq 4\pi$

C. Domain: $-\frac{2}{3} \leq x \leq \frac{2}{3}$

Range: $0 \leq y \leq \frac{\pi}{4}$

D. Domain: $-\frac{3}{2} \leq x \leq \frac{3}{2}$

Range: $0 \leq y \leq 4\pi$

7. $\int \sec^2\theta \tan^2\theta \, d\theta$ is

A. $\sec^2\theta + \frac{1}{2}\tan^2\theta + c$

B. $\frac{1}{3}\tan^3\theta + c$

C. $\frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + c$

D. $\tan^4\theta - \ln|\cos^4\theta| + c$

8. A committee of 5 people is to be chosen from a group of 6 girls and 4 boys. How many different committees could be formed that have at least one boy?

A. $\binom{10}{5} - 1$

B. $\binom{4}{1} \times \binom{6}{4}$

C. $\binom{4}{1} + \binom{6}{4}$

D. $\binom{10}{5} - 6$

9. Given that n students are to be put into 4 classrooms, what is the minimum number of students to guarantee that there are at least two students born in the same month in each of the 4 classrooms?

A. 48 B. 49 C. 52 D. 53

10. Given the parametric equations $\begin{cases} x = \sin^{-1} t + 1 \\ y = \dfrac{1}{\sqrt{1 - t^2}} \end{cases}$

A. $y = \dfrac{1}{1 - \sin(x - 1)}$

B. $y = \dfrac{1}{\sqrt{2\sin x - \sin^2 x}}$

C. $y = \dfrac{1}{\cos(x - 1)}$

D. $y = \pm\dfrac{1}{\cos(x - 1)}$

Section II

60 marks
Attempt all questions
Allow about 1 hour and 45 minutes for this section

Start each question on a new page in the answer booklet provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is available on request.

Question 11 (14 marks)

a. Solve $\dfrac{3}{3x-1} \leq 5$ [3]

b. Evaluate $\lim_{x \to 0} \left(\dfrac{\sin \frac{x}{3}}{4x} \right)$ [1]

c. Using the substitution $u = \dfrac{1}{x}$, find the exact value of

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\, dx$$ [3]

d. Evaluate $\displaystyle\int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\, dx$ using the substitution $u = x^2 + 4$ [4]

e. Use sums to products of trigonometric ratios to solve the equation
$\cos x + \cos 2x + \cos 3x = 0,\ 0 \leq x \leq 2\pi$ [3]

Question 12 (12 marks)

a. (i) Express $3\cos x + 2\sin x$ in the form $R\cos(x - \alpha)$,

where $R > 0$ and $0° < \alpha < 90°$ [3]

(ii) Hence, solve $3\cos x + 2\sin x = \sqrt{13}$, $0° \leq x \leq 360°$.

Give your answer in degrees correct to two decimal places. [2]

b. Prove by mathematical induction that

$2^{2n} + 6n - 1$ is divisible by 3 for all integers $n \geq 1$. [3]

c. Consider the function $f(x) = \sqrt{2x - 1} + 1$

(i) Find the equation of the inverse function $f^{-1}(x)$, stating its domain. [2]

(ii) Find the x – coordinates of the points where the graphs of

$y = f(x)$ and $y = f^{-1}$ intersect. [2]

Question 13 (14 marks)

a. Find $\displaystyle\int_0^{\frac{4}{3}} \frac{dx}{16 + 9x^2}$ [3]

b. (i) Show that $\displaystyle\int_0^{\frac{3\pi}{2}} \sin^2 x \, dx = \frac{3\pi}{4}$ [2]

(ii) The graph of $y = 1 + \sin x$ for $0 \leq x \leq \frac{3\pi}{2}$ is rotated

about the x – axis. Find the exact volume of the solid generated. [2]

c. A class of fifteen Mathematics students are seated around a circular table to discuss Mathematical problems.

(i) In how many different ways can the students be arranged? [1]

(ii) If the seats are randomly assigned, find the probability that four particular students, Will, Mike, Dustin and Lucas are not all seated together as a group of four? [2]

d. Consider the polynomial $P(x) = 4x^3 - 12x^2 + 5x + 6$

(i) Use the factor theorem to show that $(x - 2)$ is a factor of $P(x)$. [1]

(ii) Factorise $P(x)$ completely. [3]

Question 14 (10 marks)

a. Given that $\underset{\sim}{a} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, find

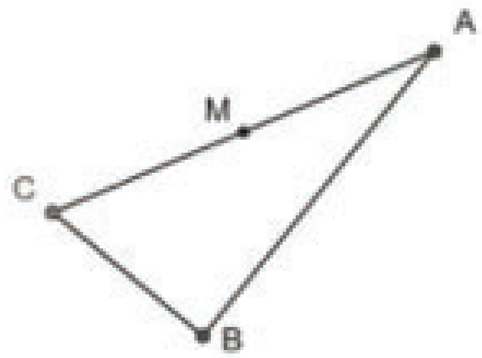
(i) $|\underset{\sim}{a}|$ [1]

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b}$ [1]

(iii) The exact angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ [2]

(iv) $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a}$ [2]

b. ΔABC is a right – angled triangle with M being the midpoint of the hypotenuse AC, as shown below. Let $\overrightarrow{AM} = \underset{\sim}{a}$ and $\overrightarrow{BM} = \underset{\sim}{b}$.



(i) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$. [2]

(ii) Prove that M is equidistant from the three vertices of ΔABC. [2]

Question 15 (10 marks)

a. A heated metal ball is dropped into a liquid.
As the ball cools, it's temperature, $T°\,C$, t minutes after it enters the liquid is given by:

$$T = 25 + 400\,e^{-0.05t}, \; t \geq 0$$

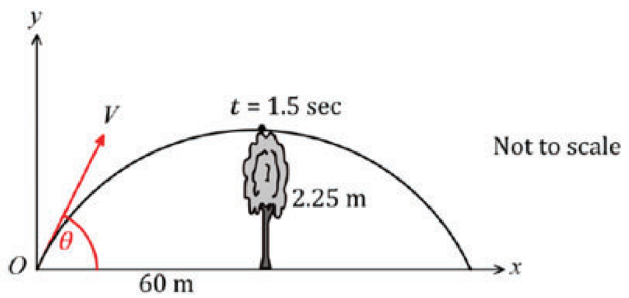
(i) Find the temperature of the ball as it enters the liquid. [1]

(ii) Find the value of t when $T = 300$.
Answer correct to 3 significant figures. [1]

(iii) Find the rate at which the temperature of the ball is decreasing at the instant when $t = 50$.
Give your answer in $°C$ per minute to 3 significant figures. [2]

(iv) Using the equation given above, explain why the temperature of the ball can never fall to $20°\,C$ [1]

b. A golfer hits a golf ball from a point O with speed V metres per second at an angle of θ above the horizontal, where $0 \leq \theta \leq \frac{\pi}{2}$.
The ball just clears a 2.25 metre high tree after 1.5 seconds.
The tree is 60 metres away from the point from which the ball was hit.
Assume $g = 10ms^{-2}$.



(i) Show that the horizontal and vertical displacement of the ball are given by
$x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$. [2]

(ii) Find the angle of projection of the ball to the nearest minute. [2]

(iii) Find the initial speed of the ball. [1]

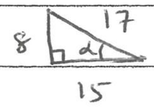
End of Examination

2022 GHS EXT 1 TRIAL SOLUTIONS

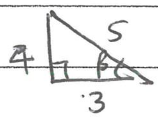
MC

1. B 2. D 3. A 4. A 5. D 6. B 7. B 8. D 9. C 10. C

1. $\sin\alpha = \frac{8}{17}$ $\sin\beta = \frac{4}{5}$



$\cos\alpha = \frac{15}{17}$



$\cos\beta = \frac{3}{5}$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \frac{8}{17}\times\frac{3}{5} + \frac{15}{17}\times\frac{4}{5}$$
$$= \frac{84}{85} \qquad [B]$$

2. $\hat{\underset{\sim}{u}} = \frac{u}{|\underset{\sim}{u}|}$

$$= \frac{2\underset{\sim}{i} - 3\underset{\sim}{j}}{\sqrt{2^2+3^2}} \qquad [D]$$
$$= \frac{1}{\sqrt{13}}(2\underset{\sim}{i} + 3\underset{\sim}{j})$$

3. $P(x) = (x+2)^3 Q(x)$

$$P'(x) = 3(x+2)^2 Q(x) + (x+2)^3 Q'(x)$$
$$= (x+2)^2[3Q(x) + (x+2)Q'(x)]$$

$\therefore (x+2)^2$ is a factor of $P'(x)$. [A]

4. $\int \sin^2 4x\, dx = \frac{1}{2}\int(1-\cos 8x)\,dx$

$$= \frac{1}{2}\left(x - \frac{1}{8}\sin 8x\right) + c$$
$$= \frac{x}{2} - \frac{1}{16}\sin 8x + c \qquad [A]$$

5. $x^3 - 2x^2 - 4x + 8 = 0$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

$$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$

$$= \frac{c/a}{-d/a}$$

$$= \frac{-4}{-8}$$

$$= \frac{1}{2} \qquad [D]$$

6. D: $-1 \le \frac{3x}{2} \le 1$

$$-\frac{2}{3} \le x \le \frac{2}{3}$$

R: $0 \le \frac{y}{4} \le \pi$

$$0 \le y \le 4\pi \qquad [B]$$

7. $\int \sec^2\theta \; \tan^2\theta \, d\theta$

$u = \tan\theta$

$du = \sec^2\theta \, d\theta$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{1}{3}\tan^3\theta + C \qquad [B]$$

8. At least 1 boy = Total − No boys

$= {}^{10}C_5 - {}^{6}C_5$

$= \binom{10}{5} - 6$ [D]

9. worst case scenario → there are 12 students all born in different months in each room ⇒ 48

One more student in each room will gurantee that there are at least 2 students in each classroom who are born in the same month.

∴ A minimum of 52 students required.

[C]

10. $\begin{cases} x = \sin^{-1} t + 1 \\ y = \dfrac{1}{\sqrt{1-t^2}} \end{cases}$ $\Rightarrow x - 1 = \sin^{-1} t$

$\sin(x-1) = t$

$\therefore y = \dfrac{1}{\sqrt{1-(\sin(x-1)^2)}}$

$= \dfrac{1}{\sqrt{\cos^2(x-1)}}$

$= \dfrac{1}{\cos(x-1)}$ [C]

Section II

Question 11 (14 marks)

a) $\dfrac{3}{3x-1} \leq 5$

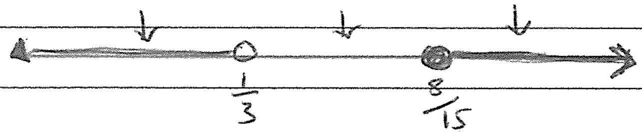
$x \neq \dfrac{1}{3}$

Solve $\dfrac{3.}{3x-1} = 5$

$5(3x-1) = 3$

$3x - 1 = \dfrac{3}{5}$

$x = \dfrac{8}{15}$



Test $x = 0$

$\dfrac{3}{-1} \leq 5$

✓

Test $x = \frac{1}{2}$

$\dfrac{3}{1/2} \leq 5$

$6 \leq 5$

×

Test $x = 1$

$\dfrac{3}{2} \leq 5$

✓

$x < \dfrac{1}{3}$; $x \geq \dfrac{8}{15}$

b) $\displaystyle\lim_{x \to 0} \left(\frac{\sin \frac{x}{3}}{4x} \right)$

$= \displaystyle\lim_{x \to 0} \left(\frac{\sin x/3}{12(x/3)} \right)$

$= \dfrac{1}{12} \displaystyle\lim_{x \to 0} \left(\frac{\sin \frac{x}{3}}{x/3} \right)$

$= \dfrac{1}{12}$

11 c) $$\int_1^2 \frac{e^{1/2}}{x^2}\,dx$$

$$u = \frac{1}{x} = x^{-1}$$

$$= -\int_1^2 e^u\,du$$

$$\frac{du}{dx} = -x^{-2} = \frac{-1}{x^2}$$

$$du = -\frac{1}{x^2}\,dx$$

$$= -\left[e^u\right]_1^{1/2}$$

when $x = 2$, $u = \frac{1}{2}$

$x = 1$, $u = 1$

$$= e - e^{1/2}$$

d) $$\int_0^{\sqrt{5}} \frac{2x^3}{\sqrt{x^2+4}}\,dx$$

$u = x^2 + 4 \Rightarrow x^2 = u - 4$

$du = 2x\,dx$

$$= \int_4^9 \frac{u-4}{\sqrt{u}}\,du$$

when $x = 0$, $u = 4$

$x = \sqrt{5}$, $u = 9$

$$= \int_4^9 \left(u^{1/2} - 4u^{-1/2}\right) du$$

$$= \left[\frac{2}{3}u^{3/2} - 8u^{1/2}\right]_4^9$$

$$= \left(\frac{2}{3}(9)^{3/2} - 8(9)^{1/2}\right) - \left(\frac{2}{3}(4)^{3/2} - 8(4)^{1/2}\right)$$

$$= \frac{14}{3}$$

11 e) $\cos x + \cos 2x + \cos 3x = 0$; $0 \le x \le 2\pi$

(using $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$\cos x + \cos 3x = 2\cos 2x \cos x$)

$\therefore$ $\cos x + \cos 2x + \cos 3x = 0$

$\Rightarrow \cos 2x + 2\cos 2x \cos x = 0$

$\cos 2x\,(1 + 2\cos x) = 0$ $(0 \le 2x \le 2\pi)$

$\cos 2x = 0$ $\cos x = -\frac{1}{2}$

$2x = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2$ $x = 2\pi/3,\ 4\pi/3$

$\therefore$ $x = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4,\ 2\pi/3,\ 4\pi/3$

Question 12

a) i) $3\cos x + 2\sin x = R\cos(x-\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

Equating coefficients,

$R\cos\alpha = 3$ — ① ; $R\sin\alpha = 2$ — ②

squaring and adding ① and ②

$R^2\cos^2\alpha + R^2\sin^2\alpha = 9+4$

$R^2(\cos^2\alpha + \sin^2\alpha) = 13$

$R = \sqrt{13}$, $R>0$

$\therefore \cos\alpha = \frac{3}{\sqrt{13}}$, $\sin\alpha = \frac{2}{\sqrt{13}}$ $\therefore 0^\circ < \alpha < 90^\circ$

$\alpha = 33.69^\circ$ (to 2dp)

$\therefore 3\cos x + 2\sin x = \sqrt{13}\cos(x - 33.69^\circ)$

(ii) $3\cos x + 2\sin x = \sqrt{13}$

$\Rightarrow \sqrt{13}\cos(x - 33.69^\circ) = \sqrt{13}$

$\cos(x - 33.69^\circ) = 1$

$-33.69^\circ \leq x - 33.69^\circ \leq 326.31^\circ$

$\therefore x - 33.69^\circ = 0^\circ$

$\therefore x = 33.69^\circ$

12 b) $2^{2n} + 6n - 1$ is divisible by 3, $n \geq 1$

Step 1 : Show true for $n = 1$

$2^{2(1)} + 6(1) - 1$

$= 9$ which is divisible by 3

Step 2 : Assume true for $n = k$

ie. $2^{2k} + 6k - 1 = 3P$ where P is an integer.

$2^{2k} = 3P - 6k + 1$

Step 3 : Prove true for $n = k+1$

ie. $2^{2(k+1)} + 6(k+1) - 1 = 3Q$ where Q is an integer.

$LHS = 2^{2(k+1)} + 6(k+1) - 1$

$= 2^{2k+2} + 6k + 6 - 1$

$= 2^2 \cdot 2^{2k} + 6k + 5$

$= 4(3P - 6k + 1) + 6k + 5$ (from assumption)

$= 12P - 24k + 4 + 6k + 5$

$= 12P - 18k + 9$

$= 3(4P - 6k + 3)$

$= 3Q$.

$= RHS$

$\therefore$ If the result is true for $n = k$, then also true for $n = k+1$

Step 4 : By the principle of mathematical induction, the result is true for all integers $n \geq 1$.

12 c) $f(x) = \sqrt{2x-1} + 1$ $\quad \left(\begin{array}{l} D: x \geq \frac{1}{2} \\ R: y \geq 1 \end{array} \right)$

i) $f(x) : y = \sqrt{2x-1} + 1$

$f^{-1}(x) : \quad x = \sqrt{2y-1} + 1$

$(x-1)^2 = 2y - 1$

$2y = (x-1)^2 + 1$

$y = \frac{1}{2}\left[(x-1)^2 + 1\right]$

Domain : $x \geq 1$

ii) At the points of intersection

$\sqrt{2x-1} + 1 = x$

$2x - 1 = (x-1)^2$

$2x - 1 = x^2 - 2x + 1$

$x^2 - 4x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{4^2 - 4(1)(2)}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= 2 \pm \sqrt{2}$$

$\therefore$ x-cordinate of point of intersection

$x = 2 + \sqrt{2}$ (since $x \geq 1$)

Question 13 (14 marks)

a) $\displaystyle\int_0^{4/3} \frac{dx}{16+9x^2}$

$= \dfrac{1}{3}\displaystyle\int_0^{4/3} \frac{3\,dx}{4^2+(3x)^2}$

$= \dfrac{1}{3} \times \dfrac{1}{4}\left[\tan^{-1}\dfrac{3x}{4}\right]_0^{4/3}$

$= \dfrac{1}{12}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$= \dfrac{\pi}{48}$

b) (i) $\displaystyle\int_0^{3\pi/2} \sin^2 x\,dx$

$= \dfrac{1}{2}\displaystyle\int_0^{3\pi/2} (1-\cos 2x)\,dx$

$= \dfrac{1}{2}\left[x - \dfrac{1}{2}\sin 2x\right]_0^{3\pi/2}$

$= \dfrac{1}{2}\left[\left(\dfrac{3\pi}{2} - \dfrac{1}{2}\sin 3\pi\right) - 0\right]$

$= \dfrac{3\pi}{4}$

(11)

b (ii)

$$V = \pi \int_0^{3\pi/2} (1+\sin x)^2 \, dx$$

$$= \pi \int_0^{3\pi/2} (1 + 2\sin x + \sin^2 x) \, dx$$

$$= \pi \left[\int_0^{3\pi/2} (1 + 2\sin x) \, dx + \int_0^{3\pi/2} \sin^2 x \, dx \right]$$

$$= \pi \left[x - 2\cos x \right]_0^{3\pi/2} + \pi \left(\frac{3\pi}{4} \right)$$

$$= \pi \left(\frac{3\pi}{2} + 2 \right) + \frac{3\pi^2}{4}$$

$$= \frac{3\pi^2}{2} + 2\pi + \frac{3\pi^2}{4}$$

$$= \left(\frac{9\pi^2}{4} + 2\pi \right) \text{ cubic units}$$

c) i) 15 students

ways of arranging in circle $= (15-1)!$

$= 14!$

ii) Ways in which the 4 students don't sit together

= total ways − no. of ways they sit together

No of ways they sit together:

12 groups $\Rightarrow (12-1)! \times 4!$

$\therefore$ Ways the sit apart $= 14! - 11!\,4!$

Probability they sit apart $= \dfrac{14! - 11!\,4!}{14!}$

$$= \frac{11!\,(14 \times 13 \times 12 - 4!)}{14!}$$

$$= \frac{90}{91}$$

13 d) $P(x) = 4x^3 - 12x^2 + 5x + 6$

i) $P(2) = 4(2)^3 - 12(2)^2 + 5(2) + 6$

$= 32 - 48 + 10 + 6$

$= 0$

$\therefore$ $(x-2)$ is a factor of $P(x)$

ii)

$$
\begin{array}{r}
4x^2 - 4x - 3 \\
x - 2 \,\big)\, \overline{4x^3 - 12x^2 + 5x + 6} \\
-\ \underline{4x^3 - 8x^2} \\
-4x^2 + 5x \\
-\ \underline{-4x^2 + 8x} \\
-3x + 6 \\
\underline{-3x + 6} \\
\cdot\cdot
\end{array}
$$

$\therefore P(x) = (x-2)(4x^2-4x-3)$

$= (x-2)(2x+1)(2x-3)$

Question 14 (10 marks)

a) $\underset{\sim}{a} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

i) $|\underset{\sim}{a}| = \sqrt{(-4)^2 + (1)^2}$

$= \sqrt{17}$

ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = (-4) \times (-3) + (1) \times (5)$

$= 17$

iii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \cdot |\underset{\sim}{b}| \cos \theta$

$\cos \theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{b}|}$

$|\underset{\sim}{b}| = \sqrt{(-3)^2 + (5)^2}$
$= \sqrt{34}$

$= \dfrac{17}{\sqrt{17} \times \sqrt{34}}$

$\cos\theta = \dfrac{1}{\sqrt{2}}$

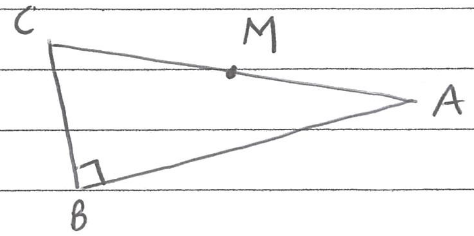
$\theta = \dfrac{\pi}{4}$

iv) $\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$

$= \dfrac{17}{34} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} -3/2 \\ 5/2 \end{bmatrix}$

14 b)



$\vec{AM} = \underset{\sim}{a}$, $\vec{BM} = \underset{\sim}{b}$

i) $\vec{AB} = \vec{AM} + \vec{MB}$
$= \vec{AM} - \vec{BM}$
$= \underset{\sim}{a} - \underset{\sim}{b}$

$\vec{BC} = \vec{BM} + \vec{MC}$
$= \vec{BM} + \vec{AM}$
$= \underset{\sim}{b} + \underset{\sim}{a}$
$= \underset{\sim}{a} + \underset{\sim}{b}$

ii) ΔABC is right angled

$\therefore \vec{AB} \perp \vec{BC}$
$\Rightarrow \vec{AB} \cdot \vec{BC} = 0$
ie. $(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = 0$
$\Rightarrow \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{b} = 0$

$|\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2 = 0$

$|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$\therefore |\vec{AM}| = |\vec{BM}| = |\vec{CM}|$

ie. M is equidistant from the three vertices of ΔABC.

Question 15 (10 marks)

a) $T = 25 + 400e^{-0.05t}$, $t \geq 0$

i) T when $t = 0$ (enters liquid)

$$T = 25 + 400e^{0}$$

$$= 425^{\circ}C$$

ii) $300 = 25 + 400e^{-0.05t}$

$$e^{-0.05t} = \frac{275}{400}$$

$$-0.05t = \ln\left(\frac{11}{16}\right)$$

$$t = 7.49 \text{ minutes} \quad \text{(to 3 sig figs)}$$

iii) $T = 25 + 400e^{-0.05t}$

$$\frac{dT}{dt} = -20e^{-0.05t}$$

when $t=50$ $\quad \frac{dT}{dt} = -20e^{-0.05(50)}$

$$= -1.64^{\circ}C \text{ per minute}$$

iv) $T = 25 + 400e^{-0.05t}$

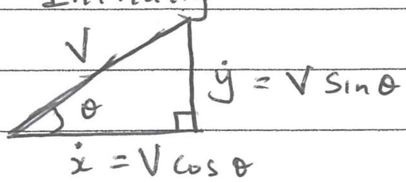
As $t \to \infty$, $\frac{400}{e^{0.05t}} \to 0$

$\therefore T \to 25 > 20$

$\therefore$ T never falls below $20^{\circ}C$

15 b) Initially

i)



Horizontally,

$\ddot{x} = 0$

$\dot{x} = C_1.$

when $t = 0$, $\dot{x} = V\cos\theta$

$\therefore C_1 = V\cos\theta$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + C_2$

when $t = 0$, $x = 0$

$\therefore C_2 = 0$

$x = Vt\cos\theta$

Vertically

$\ddot{y} = -10$

$\dot{y} = -10t + C_1$

when $t = 0$, $\dot{y} = V\sin\theta$

$\therefore C_1 = V\sin\theta$

$\dot{y} = -10t + V\sin\theta$

$y = -5t^2 + Vt\sin\theta + C_2$

when $t = 0$, $y = 0$

$\therefore C_2 = 0$

$y = -5t^2 + Vt\sin\theta$

ii) After 1.5 seconds, $x = 60$, $y = 2.25$

$60 = 1.5V\cos\theta$; $2.25 = -5(1.5)^2 + 1.5V\sin\theta$

$V\cos\theta = 40$ —① $\quad V\sin\theta = 9$ —②

① ÷ ②

$$\frac{V\sin\theta}{V\cos\theta} = \frac{9}{40}$$

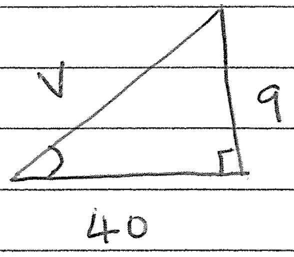
$$\tan\theta = \frac{9}{40}$$

$$\theta = 12°41'$$

The angle of projection is $12°41'$

15 b (iii) From (1) and (2) [Part (ii)]

$V \sin\theta = 9$; $V\cos\theta = 40$



$V = \sqrt{9^2 + 40^2}$

$= 41$ m/s

$\therefore$ Initial speed of the ball is 41 m/s.

End of Solutions